Week 11:

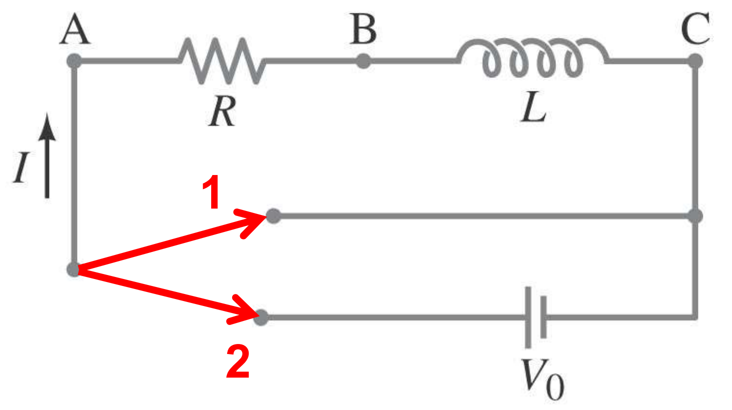
Circuits with Inductors L

LR Circuits

The sum of potential differences around the loop gives

$$V_0 - L\frac{dI}{dt} = IR \quad \rightarrow \quad L\frac{dI}{dt} + IR = V_0$$

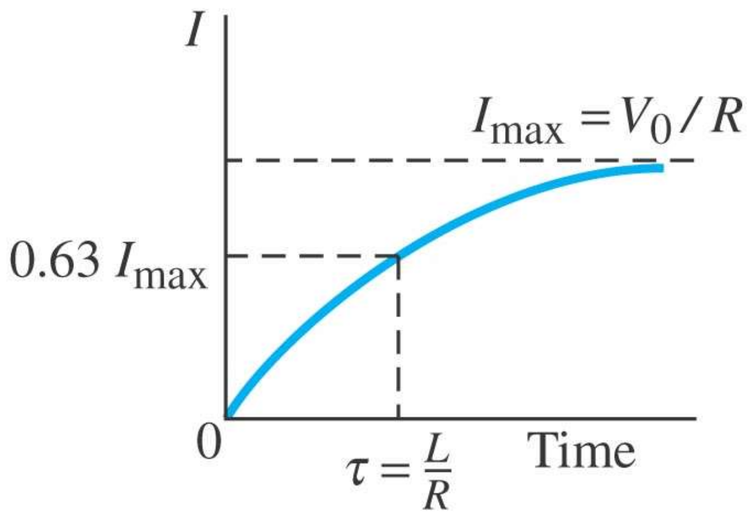
$$\frac{dI}{V_0 - IR} = \frac{dt}{L} \quad \rightarrow \quad I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$$



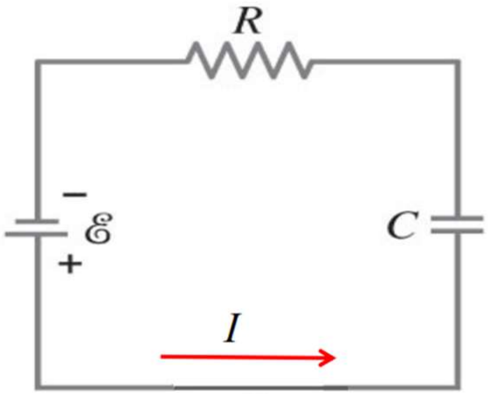
The time constant :

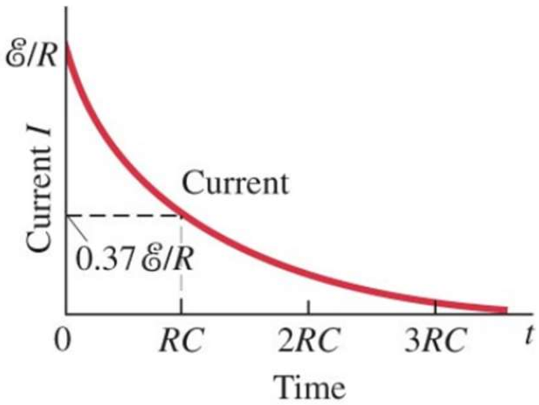
$$\tau = \frac{L}{R}$$

It represents the time required for the current I to reach or 63 % of its maximum value

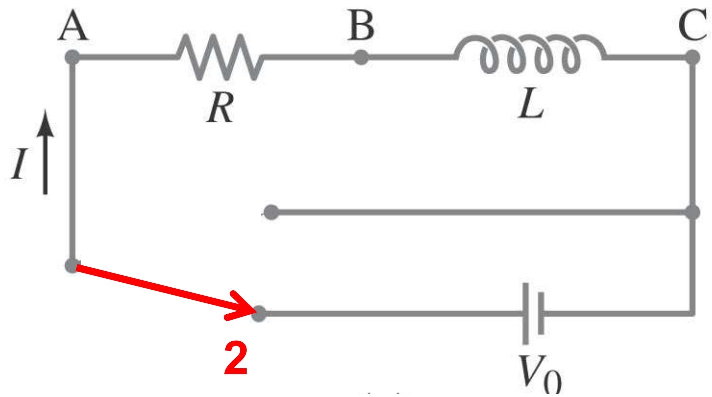


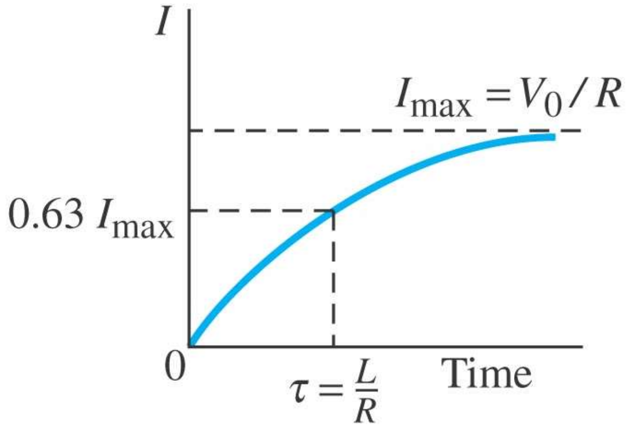
RC vs LR Circuits





$$\boxed{I = \frac{\mathcal{E}}{R} e^{-t/RC}}$$



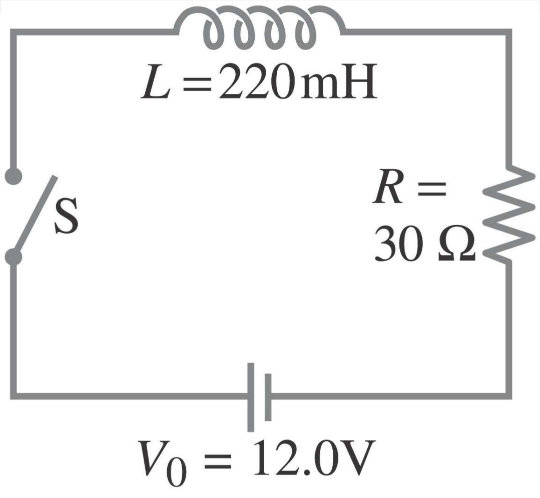


$$\boxed{I = \frac{V_0}{R}\left(1 - e^{-t \cdot R/L}\right)}$$

LR Circuits

Example: An *LR* circuit.

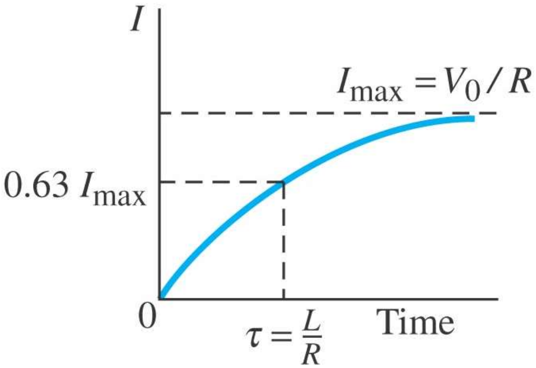
$$I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$$



(a) What is the current at $t = 0$ (just closed switch)?

$$I\Big|_{t=0} = 0$$

(b) The switch is closed. When will the maximum current be achieved and what is its magnitude?



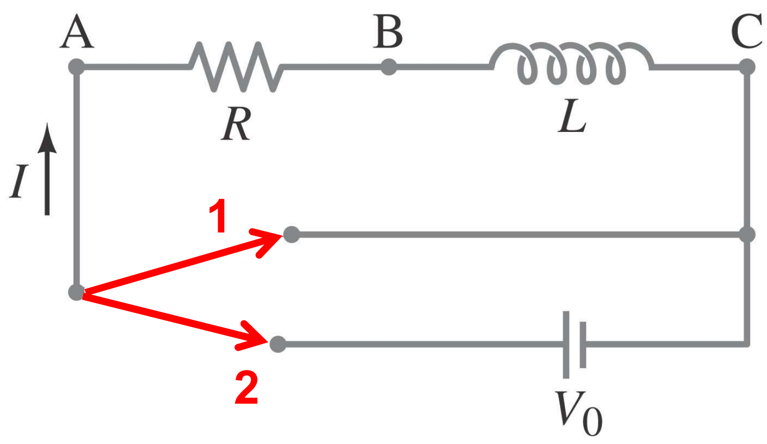
$$I_{\max} = I\Big|_{t\to\infty} = \frac{V_0}{R}$$

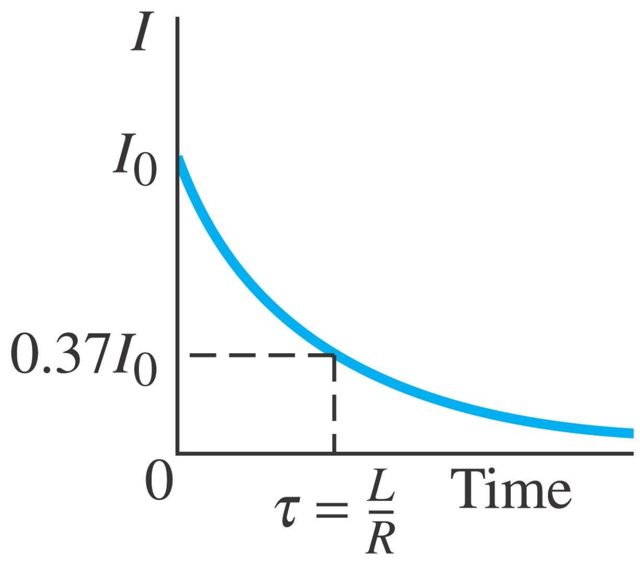
LR Circuits

If the battery is then taken out (switch position 1), the current will gradually decay away:

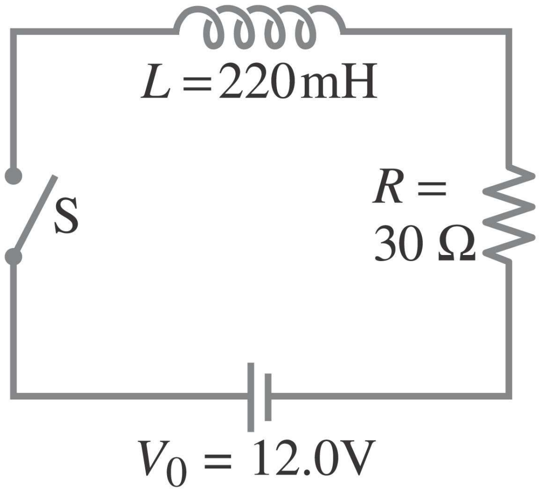
$$\cancel{V_0} - L\frac{dI}{dt} = IR \quad \rightarrow \quad \frac{dI}{I} = -\frac{R}{L}dt \quad \rightarrow \quad I = I_0 e^{-t/\tau}$$

$$\tau = \frac{L}{R}$$





LR Circuits



Example: **An *LR* circuit.**

How long will it take the current to reach half its maximum possible value?

$$I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) = \frac{1}{2}\frac{V_0}{R} \rightarrow e^{-t/\tau} = \frac{1}{2},\ t = \tau \ln 2 = \ln 2 \frac{L}{R}$$

At this instant, at what rate is energy being delivered by the battery?

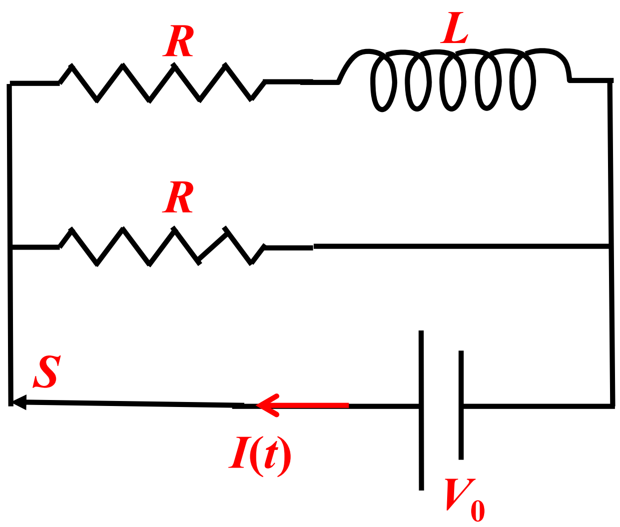
$$P = IV = \frac{V_0}{2R}V_0 = \frac{V_0^2}{2R}$$

LR Circuit Question 1

The switch S is closed and current flows.

The initial current, immediately after the switch is closed, is:

- V_0/R
- $2V_0/R$
- $V_0/2R$



The current through the inductance takes time to build up—it begins at zero. But the current through the other *R* starts immediately, so at *t* = 0 there is current around the lower loop only.

LR Circuit Question 2

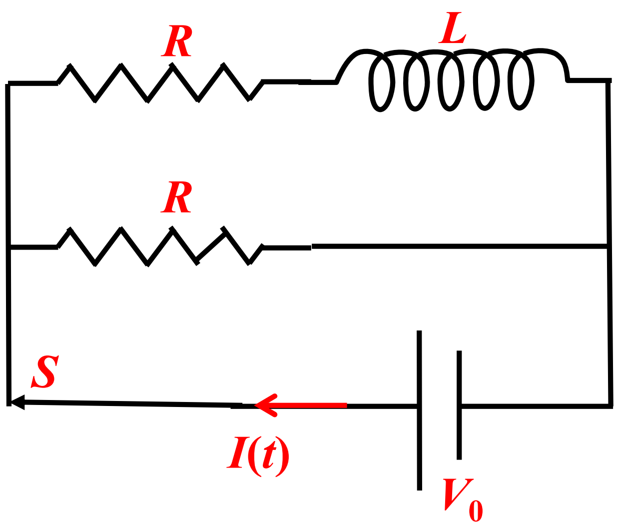
The switch S is closed and current flows.

What is the current **a long time later?**

V_0/R

$2V_0/R$

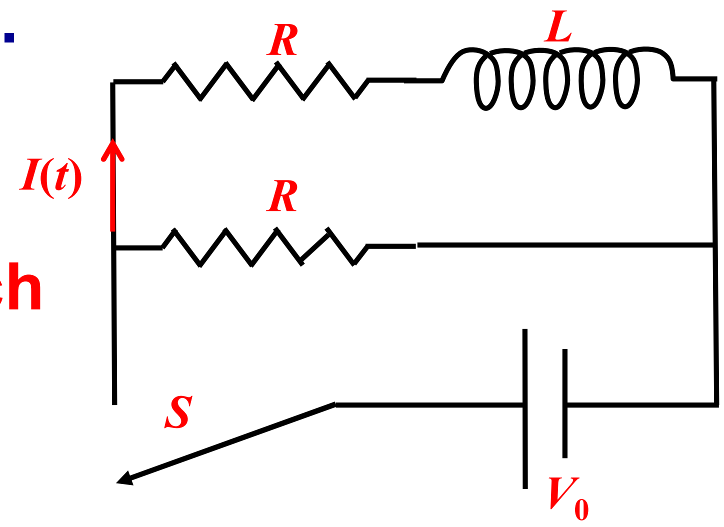
$V_0/2R$



After the current has built up to a steady value, the inductance plays no further role as long as the current remains steady.

LR Circuit Question 3

After this long time, the switch is suddenly opened.



What are the currents immediately after the switch is opened?

- V_0/R **in the upper loop**
- $2V_0/R$ **in the upper loop**
- **all currents zero**

The inductance will not allow sudden discontinuous change in current, so the current through it will be the same just after opening the switch as it was before. This current must now go back via the *other* resistance.

LC Circuits

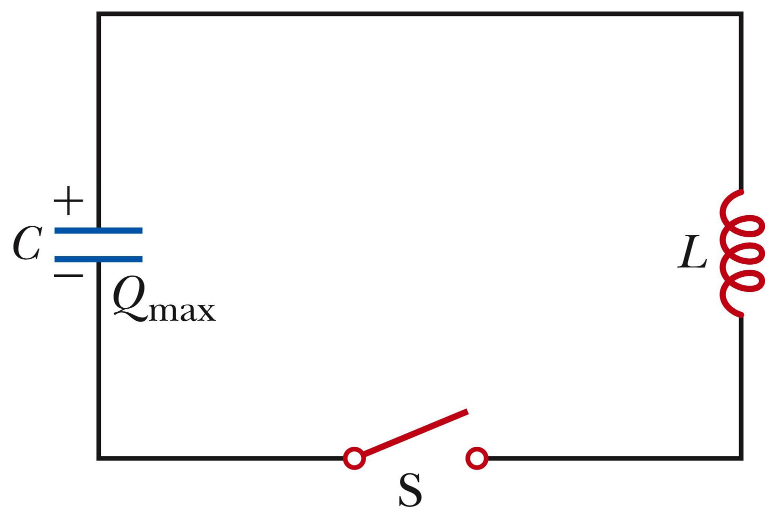
A capacitor is connected to an inductor in an LC circuit.

Assume the capacitor is initially charged and then the switch is closed.

Assume no resistance and no energy losses to radiation.

So the **total energy in the LC circuit is conserved** at every instant.

It is an energy distributed partially in the capacitor C and partially in the inductor L.



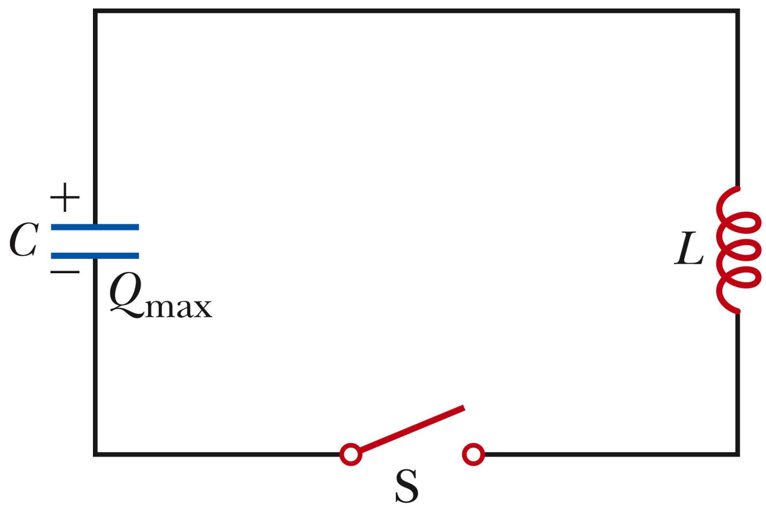
Oscillations in an *LC* Circuit

Under the previous conditions, the current in the circuit and the charge on the capacitor **oscillate between maximum positive and negative values**.

With zero resistance, no energy is transformed into internal energy.

Ideally, the oscillations in the circuit persist indefinitely.

- The idealizations are no resistance and no radiation.



STARTING POINT: The capacitor is fully charged.

- The energy U in the circuit is stored in the electric field of the capacitor.
- The energy is equal to $Q^2_{max} / 2C$.
- The current in the circuit is zero.
- No energy is stored in the inductor.

The switch is closed.

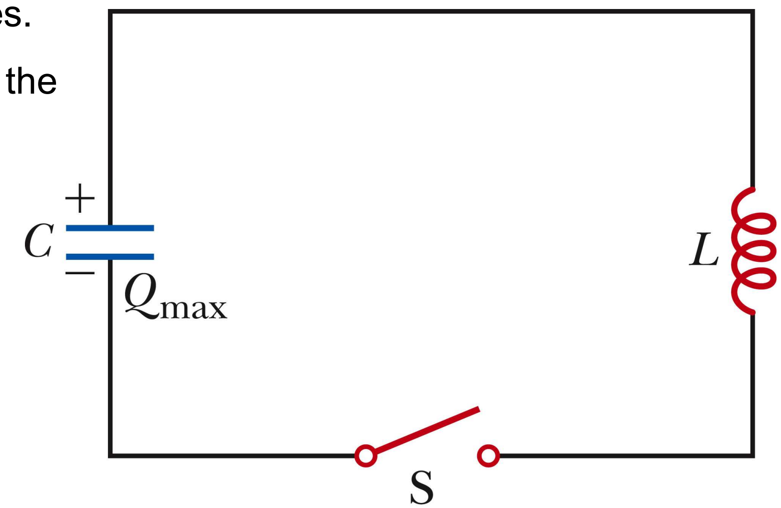
Oscillations in an *LC* Circuit

The current is equal to the rate at which the charge changes on the capacitor.

- As the capacitor discharges, the energy stored in the electric field decreases.
- Since there is now a current, some energy is stored in the magnetic field of the inductor.
- Energy is transferred from the electric field to the magnetic field.

Eventually, the capacitor becomes fully discharged.

- It stores no electric energy.
- All of the energy is stored in the magnetic field of the inductor.
- The current reaches its maximum value.



The current now decreases in magnitude, recharging the capacitor with its plates having opposite their initial polarity.

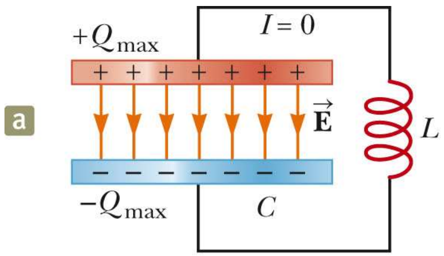
The capacitor becomes fully charged and the cycle repeats (no resistor).

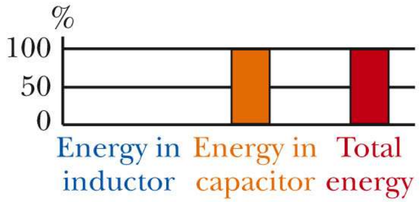
The energy continues to oscillate between the inductor and the capacitor.

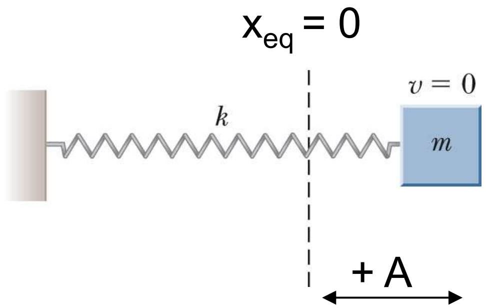
The total energy stored in the LC circuit remains constant in time and equals.

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2} L I^2$$

LC Circuit Analogy to Spring-Mass System, 1





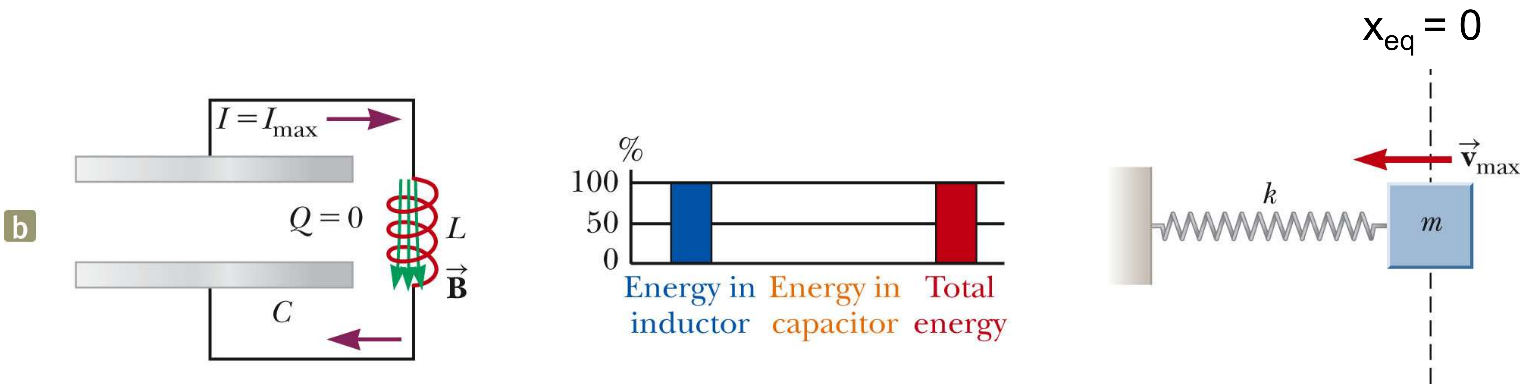


The potential energy ½kx^2 stored in the spring is analogous to the electric potential energy $(Q_{max})^2/(2C)$ stored in the capacitor.

All the energy is stored in the capacitor at t = 0.

This is analogous to the spring stretched to its maximum positive amplitude + A.

LC Circuit Analogy to Spring-Mass System, 2



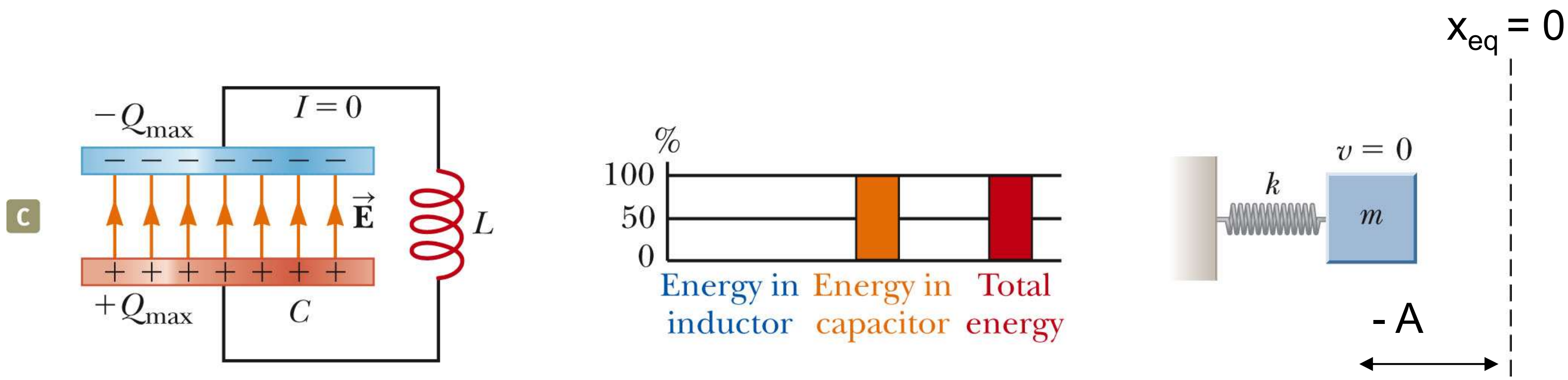
The kinetic energy (½ mv^2) of the mass m attached to the spring is analogous to the magnetic energy (½ $L I^2$) stored in the inductor.

At t = ¼ T, all the energy is stored as magnetic energy in the inductor.

The maximum current occurs in the circuit.

This is analogous to the mass at equilibrium.

LC Circuit Analogy to Spring-Mass System, 3

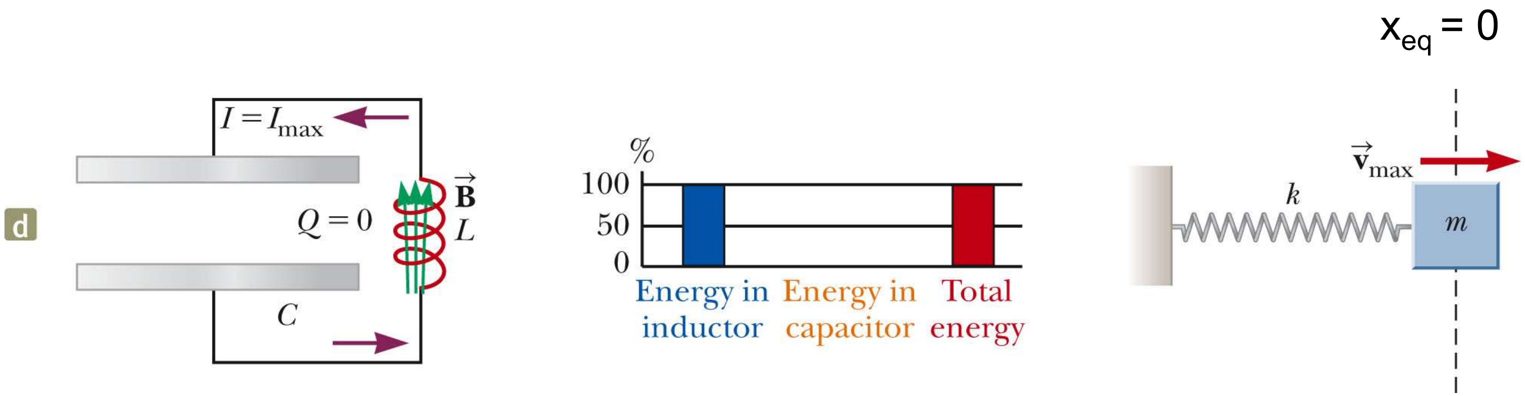


At t = ½ T, the energy in the circuit is completely stored in the capacitor.

The polarity of the capacitor is reversed.

This is analogous to the spring compressed to the maximum of its negative distance from the equilibrium position - A.

LC Circuit Analogy to Spring-Mass System, 4



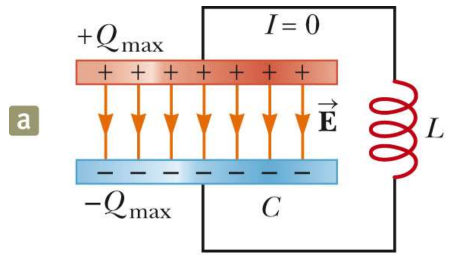
At t = ¾ T, the energy is again stored in the magnetic field of the inductor.

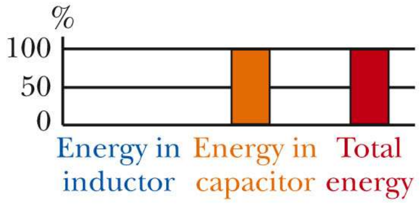
This is analogous to the mass again reaching the equilibrium position.

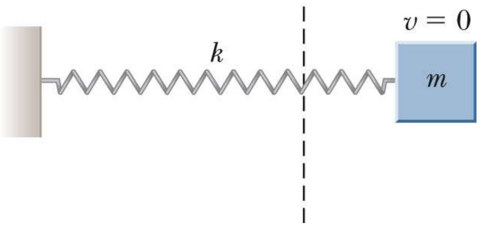
LC Circuit Analogy to Spring-Mass System, 5

At t = T, the cycle is completed

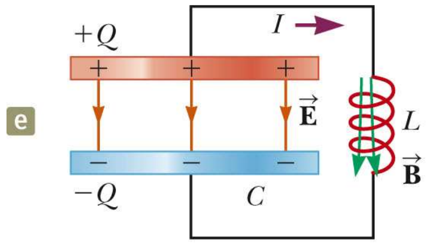
The conditions return to those identical to the initial conditions.

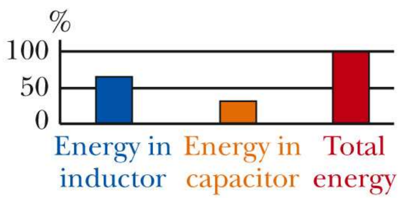


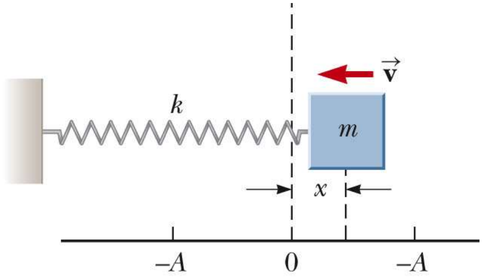




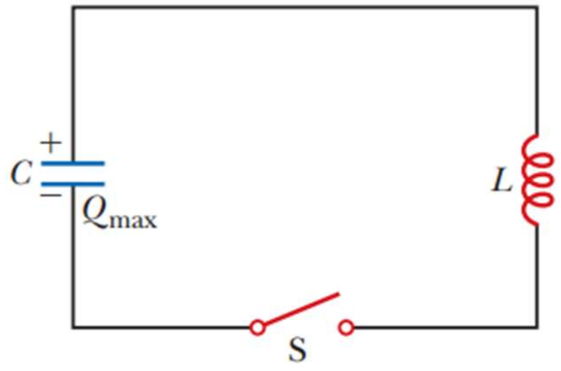
At other points in the cycle, energy is shared between the electric and magnetic fields.



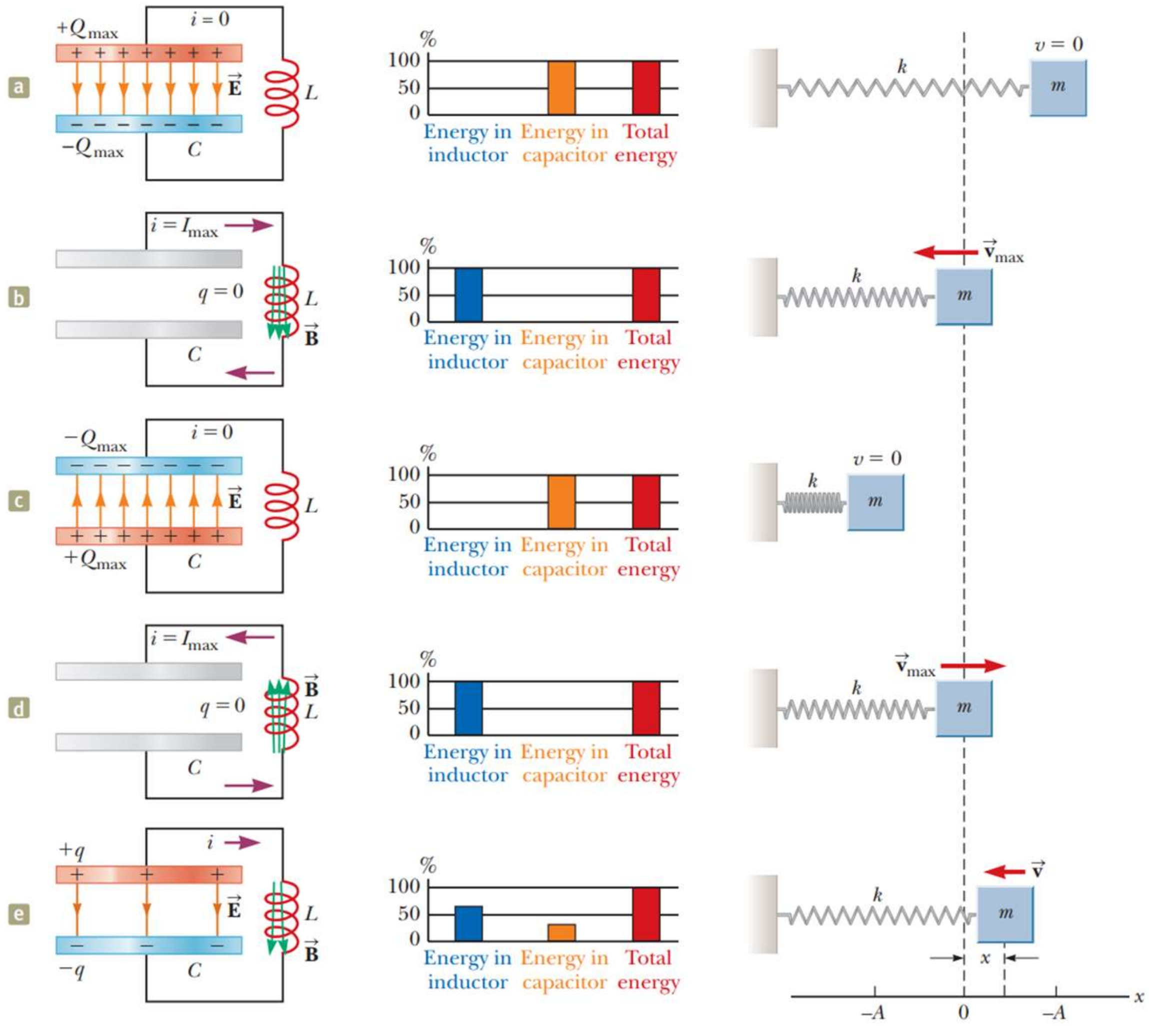




Energy oscillation in LC Circuits - summary



A simple LC circuit. The capacitor has an initial charge Q_{max}, and the switch is open for $t < 0$ and then closed at $t = 0$.



Time Functions of an *LC* Circuit

In an LC circuit, charge can be expressed as a function of time.

- $Q = Q_{max} \cos(\omega t + \varphi)$
- This is for an ideal LC circuit

The angular frequency, ω, of the circuit depends on the inductance and the capacitance.

- It is the natural frequency of oscillation of the circuit.

$$\omega = 1/\sqrt{LC}$$

The current can be expressed as a function of time:

$$I = \frac{dQ}{dt} = -\omega Q_{max} \sin(\omega t + \varphi)$$

The total energy (at some instant *t*) can be expressed as a function of time:

$$U = U_C + U_L = \frac{Q_{max}^2}{2c} \cos^2 \omega t + \frac{1}{2} L I_{max}^2 \sin^2 \omega t$$

Charge and Current in an LC Circuit

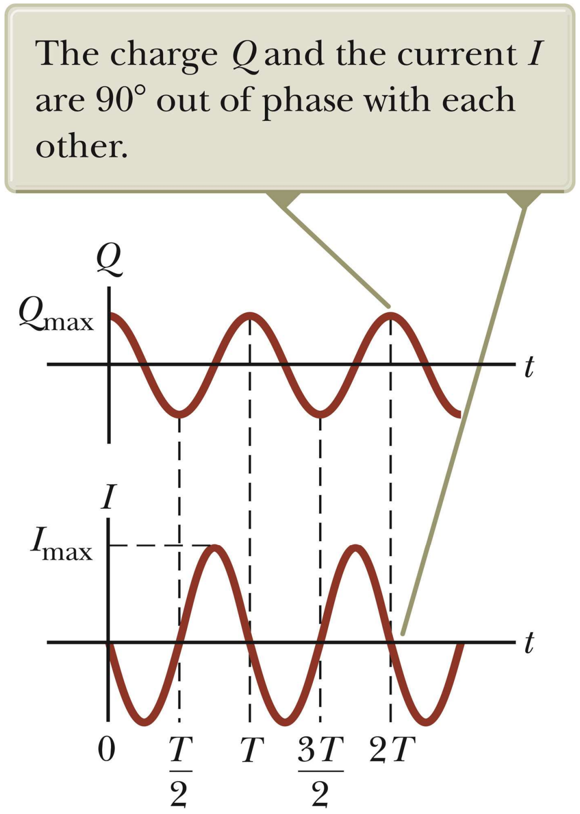
The charge on the capacitor oscillates between Q_{max} and $-Q_{max}$.

The current in the inductor oscillates between I_{max} and $-I_{max}$.

Q and I are 90° out of phase with each other

- So when Q is a maximum,

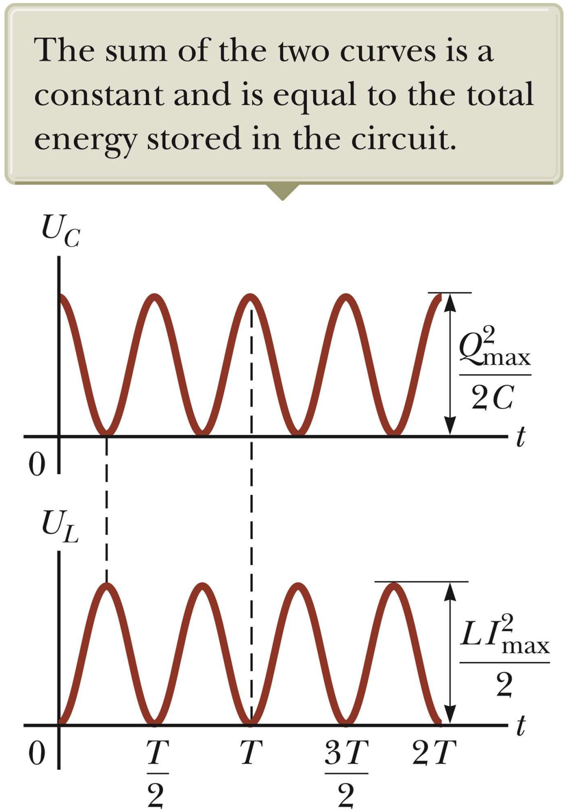
  I is zero, etc.



Energy in an LC Circuit – Graphs

The energy continually oscillates between the energy stored in the electric and magnetic fields.

When the total energy is stored in one field, the energy stored in the other field is zero.



LC Circuits - summary

$$U = U_E + U_B = \frac{Q_{max}^2}{2C}\cos^2\omega t + \tfrac{1}{2}LI_{max}^2\sin^2\omega t$$

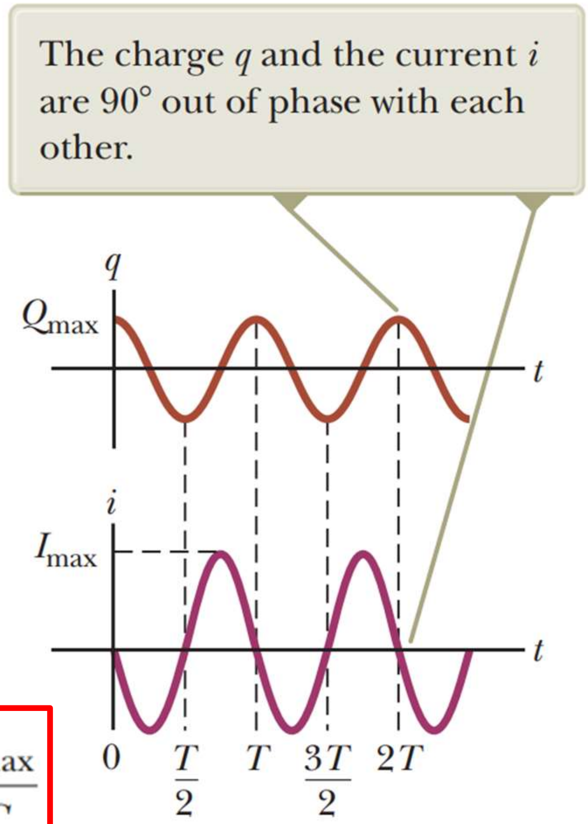
The energy of the LC circuit continuously oscillates between energy stored in the capacitor's electric field and energy stored in the inductor's magnetic field.

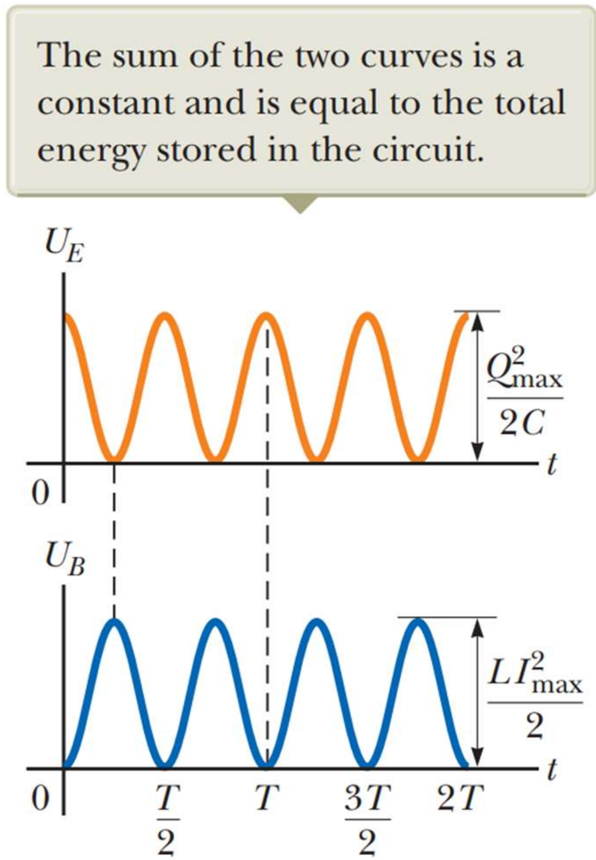
The amplitudes of the two graphs in Figure must be equal because the maximum energy stored in the capacitor (when $I = 0$) must equal the maximum energy stored in the inductor (when $q = 0$). This equality is expressed mathematically as

$$\frac{Q_{max}^2}{2C} = \frac{LI_{max}^2}{2}$$

$$\cos^2\omega t + \sin^2\omega t = 1.$$

$$\boxed{U = \frac{Q_{max}^2}{2C}(\cos^2\omega t + \sin^2\omega t) = \frac{Q_{max}^2}{2C}}$$

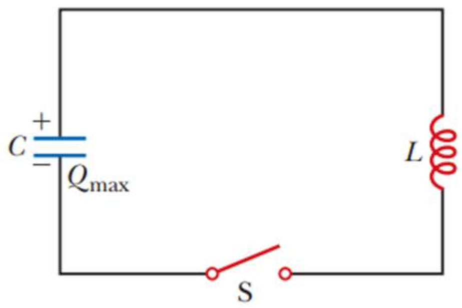




LC Circuits (formal solution)

Assume the capacitor has an initial charge Q_{max} (the maximum charge) and the switch is open for $t < 0$ and then closed at $t = 0$. Let's investigate what happens from an energy viewpoint.

When the capacitor is fully charged, the energy U in the circuit is stored in the capacitor's electric field and is equal to $Q_{max}^2/2C$. At this time, the current in the circuit is zero; therefore, no energy is stored in the inductor.



A simple LC circuit. The capacitor has an initial charge Q_{max}, and the switch is open for $t < 0$ and then closed at $t = 0$.

Consider some arbitrary time t after the switch is closed so that the capacitor has a charge $q(t) < Q_{max}$, and the current is $i(t) < I_{max}$. At this time, both circuit elements store energy, but the sum of the two energies must equal the total initial energy U stored in the fully charged capacitor at $t = 0$

$$U = U_E + U_B = \frac{q^2}{2C} + \tfrac{1}{2}Li^2 = \frac{Q_{\max}^2}{2C}$$

The total energy of the system must remain constant in time

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \tfrac{1}{2}Li^2\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

$$i = dq/dt$$
$$di/dt = d^2q/dt^2$$

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\boxed{\frac{d^2q}{dt^2} = -\frac{1}{LC}q}$$

Second order differential equation

LC **Circuits** (formal solution)

Let's solve for q by noting that this expression is of the same form for a particle in simple harmonic motion:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

where k is the spring constant, m is the mass of the block, and $\omega = \sqrt{k/m}$. The solution of this mechanical equation has the general form:

$$x = A\cos(\omega t + \phi)$$

Because Equation $\frac{d^2q}{dt^2} = -\frac{1}{LC}q$ is of the same mathematical form as the differential equation of the simple harmonic oscillator, it has the solution

$$\boxed{q = Q_{\max}\cos(\omega t + \phi)}$$

where Q_{max} is the maximum charge of the capacitor and the angular frequency ω is the natural frequency of oscillation of the LC circuit

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\boxed{i = \frac{dq}{dt} = -\omega Q_{\max}\sin(\omega t + \phi)}$$

To determine the value of the phase angle ϕ, let's examine the initial conditions, which in our situation require that at $t = 0$, $i = 0$ and $q = Q_{max}$.

$$0 = -\omega Q_{\max}\sin\phi \quad \Longrightarrow \quad \phi = 0 \quad \Longrightarrow \quad q = Q_{\max}\cos\omega t$$

Notes About **Real** *LC* Circuits

In actual circuits, there is always some resistance. → ***RLC* Circuit**

Therefore, there is some energy transformed to internal energy of the resistor (Joule heating).

Radiation is also inevitable in this type of circuit. (we will see that oscillating charges will generate radiation!)

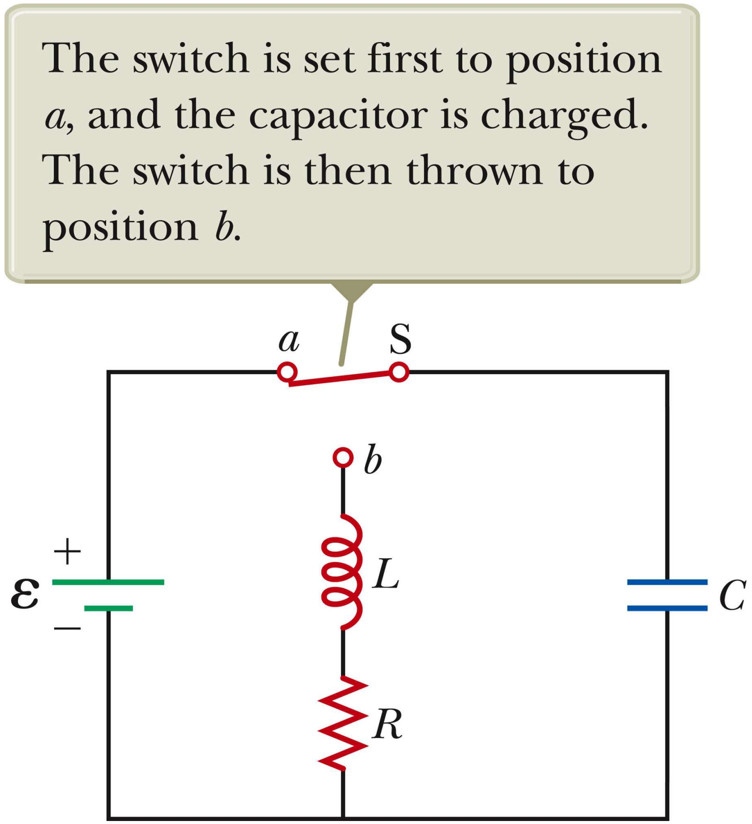
In a REAL *LC* circuit,

the total energy in the circuit continuously decreases as a result of these processes.

The *RLC* Circuit

A circuit containing a resistor, an inductor and a capacitor is called an *RLC* Circuit.

Assume the resistor represents the total resistance of the circuit.



RLC Circuit, Analysis

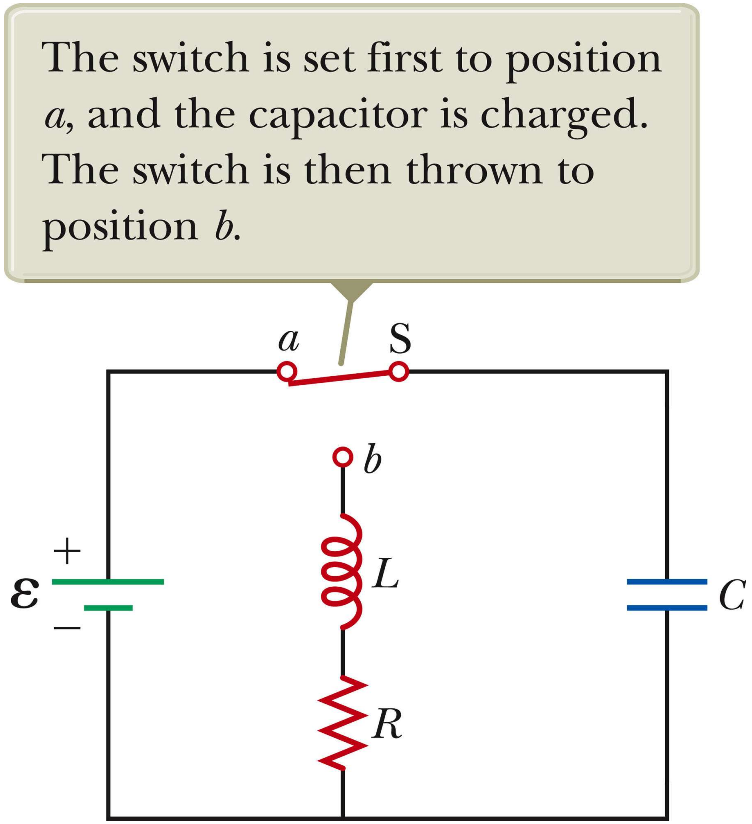
The total energy is not constant, since there is a transformation to internal energy in the resistor at the rate of $dU/dt = -I^2R$.

- Radiation losses are still ignored

The circuit's operation can be expressed as (Kirchoff)

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

Eneregy loss = damping term



RLC Circuit Compared to Damped Oscillators

The *RLC* circuit is analogous to a **damped harmonic oscillator**.

When R = 0

- The circuit reduces to an *LC* circuit and is equivalent to no damping in a mechanical oscillator.

When R is small:

- The RLC circuit is analogous to “light damping” in a mechanical oscillator.
- $Q = Q_{max}\, e^{-Rt/2L} \cos \omega_d t$
- ω_d is the angular frequency of oscillation for the circuit and

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2}$$

We will not derive this equation

RLC Circuit Compared to Damped Oscillators

When R is very large, the oscillations damp out very rapidly.

There is a "critical value" of R above which no oscillations occur.

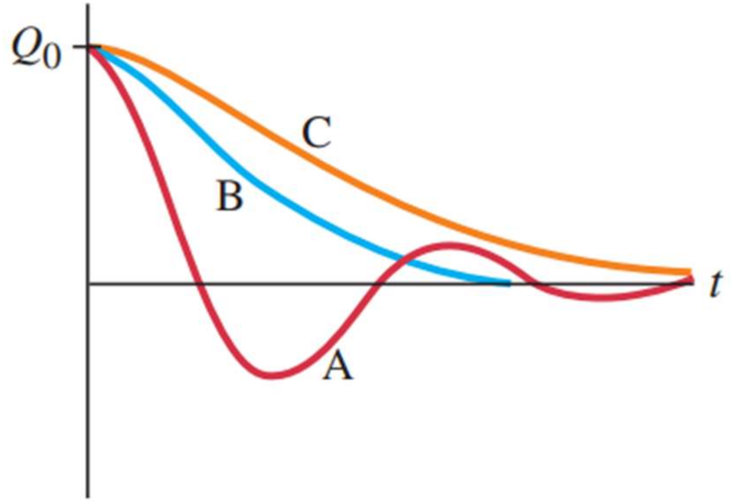
$$R_C = \sqrt{4L / C}$$

(B) If R = R_C, the circuit is said to be "***critically damped***".

(C) When R > R_C, the circuit is said to be "***overdamped***".

(A) When R < R_C, the circuit is said to be "***underdamped***".

FIGURE Charge Q on the capacitor in an LRC circuit as a function of time: curve A is for underdamped oscillation $(R^2 < 4L/C)$, curve B is for critically damped $(R^2 = 4L/C)$, and curve C is for overdamped $(R^2 > 4L/C)$.

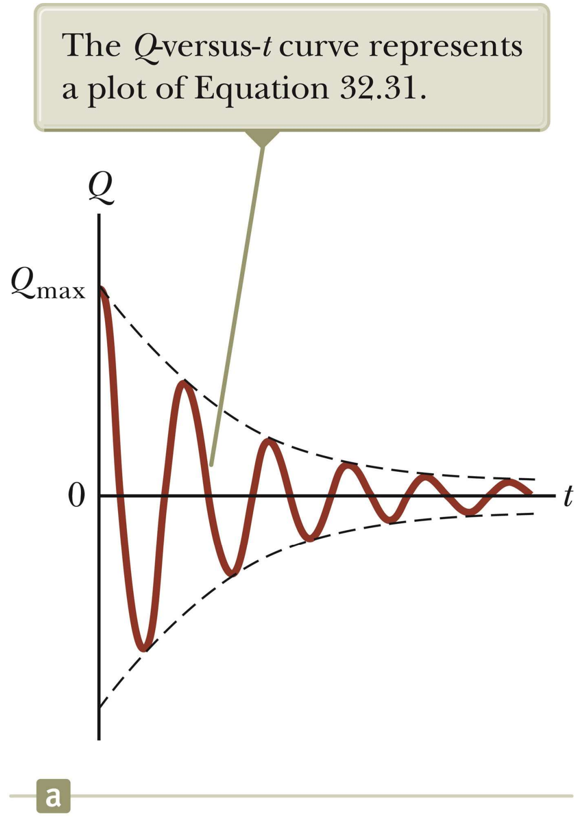


Damped *RLC* Circuit, Graph

The maximum value of *Q* decreases after each oscillation.

- $R < R_C$

This is analogous to the amplitude of a damped spring-mass system.



Infinite oscillations: LC circuit

- the oscillations are very similar to the oscillations of a pendulum:

LC circuit

$$\underbrace{L\ddot{Q}}_{\text{"kinetic" term}} + \underbrace{Q/C}_{\text{"potential" term}} = 0$$

$$\ddot{Q} + \omega^2 Q = 0, \quad \omega = \frac{1}{\sqrt{LC}}$$

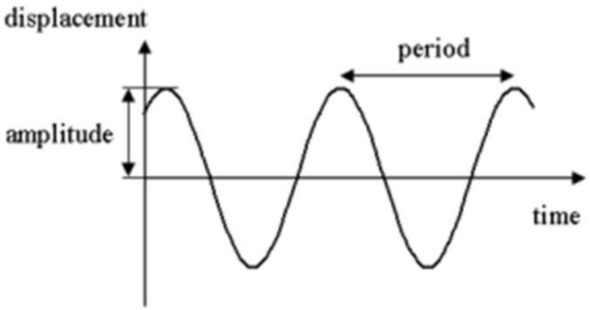
$$Q(t) = Q_0 \cos(\omega t + \phi)$$

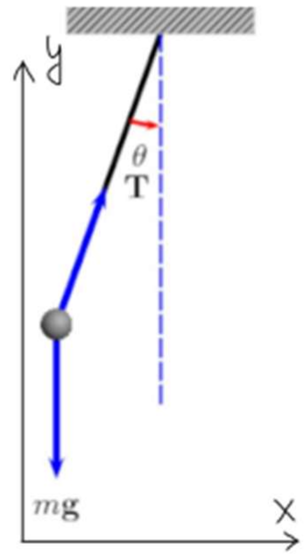
Pendulum

$$\underbrace{ml^2\ddot{\theta}}_{\text{kinetic term}} + \underbrace{mg\theta}_{\text{"potential" term}} = 0$$

$$\ddot{\theta} + \omega^2 \theta = 0, \quad \omega = \sqrt{\frac{g}{l}}$$

$$\theta(t) = \theta_0(\cos \omega t + \phi)$$





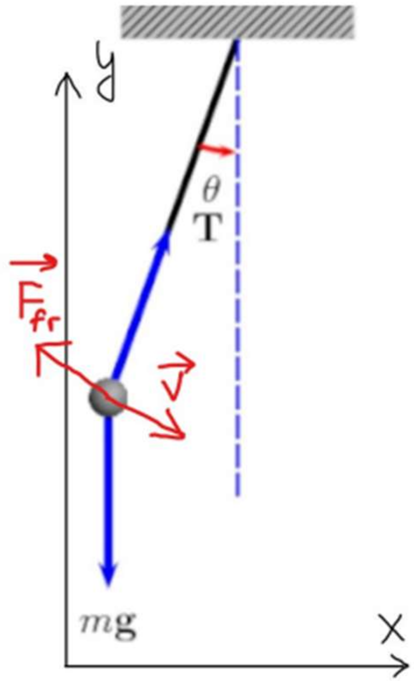
Damped oscillations: RLC circuit

- the equation $\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{LC} = 0$ has the form:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

- this is the same equation of a pendulum with *a friction* $\boldsymbol{F}_{fr} = -\eta \boldsymbol{v}$:

$$m\ell^2 \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\ell\theta - \eta\ell\frac{\mathrm{d}\theta}{\mathrm{d}t}$$



RLC circuit

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{LC}Q = 0$$

$$x(t) = Q(t), \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad \gamma = \frac{R}{2L}$$

Pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{\eta}{\ell}\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{\ell}\theta = 0$$

$$x(t) = \theta(t), \quad \omega_0 = \sqrt{\frac{g}{\ell}}, \quad \gamma = \frac{\eta}{2\ell}$$

Damped oscillations: RLC circuit

- from the analogy with the pendulum, we expect that oscillations cannot last forever and eventually decay, having lost all its energy
- if the energy losses are sufficiently low, $\gamma < \omega$, the solution is

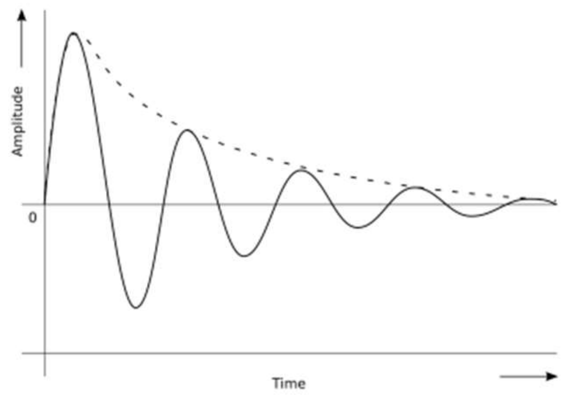
$$Q(t) = Q_0 \cdot \underbrace{e^{-\gamma t}}_{\text{decay}} \cdot \underbrace{\cos\sqrt{\omega_0^2 - \gamma^2}\,t}_{\text{oscillations}}$$

- indeed, the amplitude of oscillations decreases with exponential rate with time

- the amplitude decreases by $e^{-\gamma T}$ after each period, and the actual angular frequency becomes smaller:

$$\omega = \sqrt{\omega_0^2 - \gamma^2} < \omega_0$$

- **to maintain the oscillations and compensate the energy losses, we need to add an energy source**



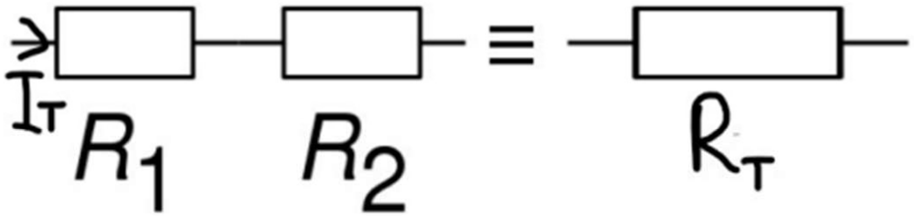
Summary: Analogies Between Electrical and Mechanic Systems

TABLE 32.1 *Analogies Between Electrical and Mechanical Systems*

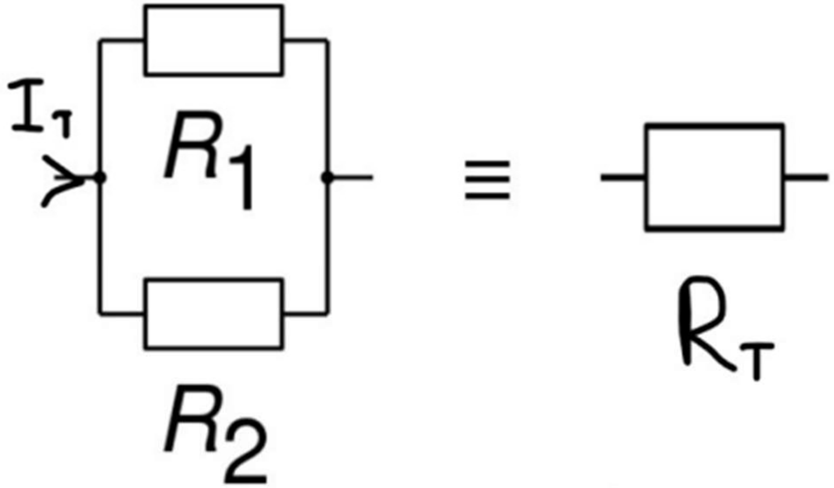
Electric Circuit		**One-Dimensional Mechanical System**
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring

REMINDER

Resistors in series and in parallel



Resistors in series



Parallel resistors

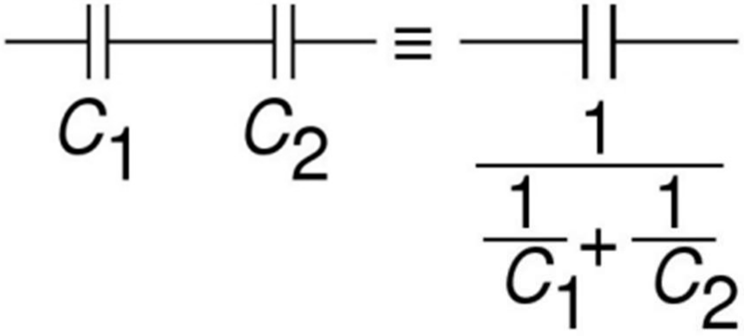
$$V_T = V_1 + V_2, \qquad I_1 = I_2 = I_T$$

$$R_T = \frac{V_T}{I_T} = \frac{V_1}{I_T} + \frac{V_2}{I_T} = R_1 + R_2$$

$$V_T = V_1 = V_2, \qquad I_T = I_1 + I_2$$

$$R_T = \frac{V_T}{I_T} = \frac{1}{\frac{I_1}{V_T} + \frac{I_2}{V_T}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

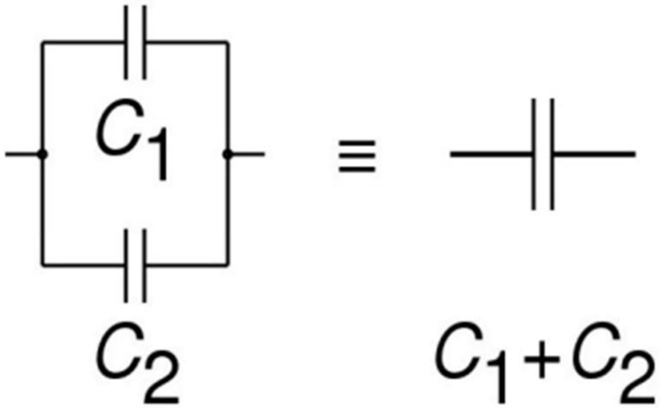
Capacitors in series and parallel



Capacitors in series

$$V_T = V_1 + V_2,$$
$$I_1 = I_2 = I_T \Rightarrow Q_1 = Q_2 = Q_T$$

$$C_T = \frac{Q_T}{V_T} = \frac{1}{\frac{V_1}{Q_T} + \frac{V_2}{Q_T}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

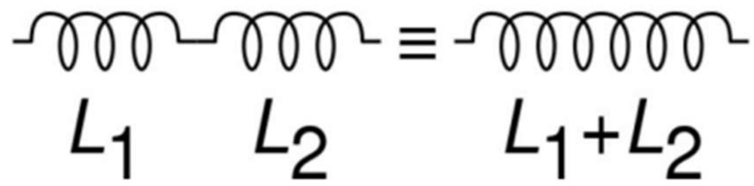


Parallel capacitors

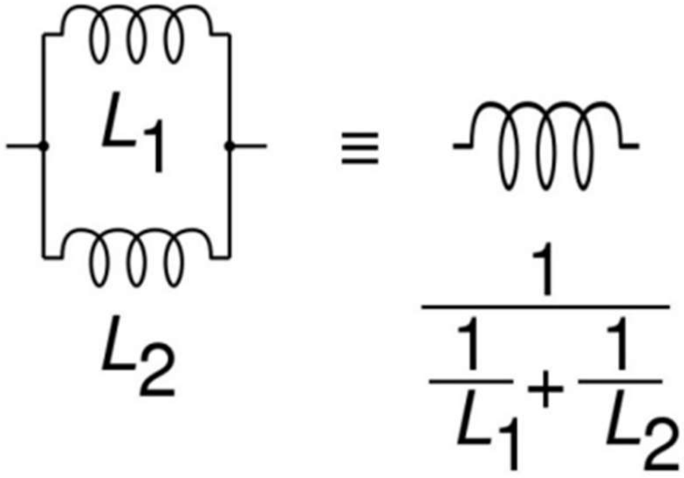
$$V_T = V_1 = V_2,$$
$$I_T = I_1 + I_2 \Rightarrow Q_T = Q_1 + Q_2$$

$$C_T = \frac{Q_T}{V_T} = \frac{Q_1}{V_T} + \frac{Q_2}{V_T} = C_1 + C_2$$

Inductance in series and parallel



Inductance in series



Parallel inductance

$$V_T = V_1 + V_2,$$

$$I_1 = I_2 = I_T \Rightarrow \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dI_T}{dt}$$

$$L_T = \frac{V_T}{dI_T/dt} = \frac{V_1}{dI_T/dt} + \frac{V_2}{dI_T/dt} = L_1 + L_2$$

$$V_T = V_1 = V_2,$$

$$I_T = I_1 + I_2 \Rightarrow \frac{dI_T}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$L_T = \frac{V_T}{dI_T/dt} = \frac{1}{\frac{1}{V_T}\frac{dI_1}{dt} + \frac{1}{V_T}\frac{dI_2}{dt}} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2}}$$